

1-Sperner hypergraphs*

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October 9, 2015

Abstract

We introduce a new class of hypergraphs, the class of 1-Sperner hypergraphs. A hypergraph \mathcal{H} is said to be 1-*Sperner* if every two distinct hyperedges e, f of \mathcal{H} satisfy $\min\{|e \setminus f|, |f \setminus e|\} = 1$. We prove a decomposition theorem for 1-Sperner hypergraphs and examine several of its consequences, including bounds on the size of 1-Sperner hypergraphs and a new, constructive proof of the fact that every 1-Sperner hypergraph is threshold. We also show that within the class of normal Sperner hypergraphs, the (generally properly nested) classes of 1-Sperner hypergraphs, of threshold hypergraphs, and of 2-asummable hypergraphs coincide. This yields new characterizations of the class of threshold graphs.

1 Introduction

A *hypergraph* \mathcal{H} is a pair (V, \mathcal{E}) where $V = V(\mathcal{H})$ is a finite set of *vertices* and $\mathcal{E} = E(\mathcal{H})$ is a set of subsets of V , called *hyperedges*. A hypergraph is said to be *Sperner* (or: a *clutter*) if no hyperedge contains another one, that is, if $e, f \in \mathcal{E}$ and $e \subseteq f$ implies $e = f$. A hypergraph is said to be *threshold* if there exist a non-negative integer weight function $w : V \rightarrow \mathbb{Z}_{\geq 0}$ and a non-negative integer threshold $t \in \mathbb{Z}_{\geq 0}$ such that for every subset $X \subseteq V$, we have

*The authors thank for support the National Science Foundation (Grant IIS-1161476) and the Slovenian Research Agency (I0-0035, research program P1-0285 and research projects BI-US/14-15-050, N1-0032, J1-5433, J1-6720, and J1-6743).

$w(X) := \sum_{x \in X} w(x) \geq t$ if and only if $e \subseteq X$ for some $e \in \mathcal{E}$. Threshold hypergraphs were defined in the uniform case by Golumbic [10] and studied further by Reiterman et al. [20]. In their full generality (that is, without the restriction that the hypergraph is uniform), the concept of threshold hypergraphs is equivalent to that of threshold monotone Boolean functions, see, e.g., [17]. A polynomial time recognition algorithm for threshold monotone Boolean functions represented by their complete DNF was given by Peled and Simeone [19]. The algorithm is based on linear programming and implies a polynomial time recognition algorithm for threshold hypergraphs.

The mapping that takes every hyperedge $e \in \mathcal{E}$ to its *characteristic vector* $\chi^e \in \{0, 1\}^V$, defined by

$$\chi_v^e = \begin{cases} 1, & \text{if } v \in e \\ 0, & \text{otherwise,} \end{cases}$$

shows that the sets of hyperedges of threshold Sperner hypergraphs are in a one-to-one correspondence with the sets of minimal feasible binary solutions to a linear inequality of the form $w^\top x \geq t$ where $w \in \mathbb{Z}_{\geq 0}^V$ and $t \in \mathbb{Z}_{\geq 0}$.

A set of vertices $X \subseteq V$ in a hypergraph is said to be *independent* if it does not contain any hyperedge, and *dependent* otherwise. Thus, threshold hypergraphs are exactly the hypergraphs admitting a linear function on the vertices separating the characteristic vectors of the independent sets from the characteristic vectors of dependent sets.

Sperner hypergraphs can be equivalently defined as the hypergraphs such that every two distinct hyperedges e and f satisfy

$$\min\{|e \setminus f|, |f \setminus e|\} \geq 1. \quad (1)$$

This point of view motivated Chiarelli and Milanić to define in [5] a hypergraph \mathcal{H} to be *dually Sperner* if every two distinct hyperedges e and f satisfy

$$\min\{|e \setminus f|, |f \setminus e|\} \leq 1. \quad (2)$$

It was shown in [5] that dually Sperner hypergraphs are threshold. In [4, 5], these hypergraphs were applied to characterize two classes of graphs defined by the following properties. Every induced subgraph has a non-negative linear vertex weight function separating the characteristic vectors of all total dominating sets [4] (resp., connected dominating sets [5]) from the characteristic vectors of all other sets.

The non-negativity of the weight function w in the definition of threshold hypergraphs implies that only minimal hyperedges matter for the thresholdness property of a given hypergraph. Since dually Sperner hypergraphs are threshold, we focus on the family of hypergraphs that are both Sperner and dually Sperner. We call such hypergraphs *1-Sperner*. Thus, a hypergraph \mathcal{H} is 1-Sperner if and only if for every pair e, f of distinct hyperedges of \mathcal{H} both inequalities (1), (2) hold. In other words, every two distinct hyperedges e and f satisfy $\min\{|e \setminus f|, |f \setminus e|\} = 1$. Note that by definition, all hypergraphs with at most one hyperedge (possibly with no vertices) are 1-Sperner.

The aim of this paper is twofold. First, we prove a decomposition theorem for 1-Sperner hypergraphs and examine several of its consequences, including the fact that the characteristic vectors of the hyperedges of a 1-Sperner hypergraph are linearly independent. From this

we derive an upper bound on the size of a 1-Sperner hypergraph, as well as a lower bound on the size of a 1-Sperner hypergraph without universal, isolated, and twin vertices. Second, we show that within the class of normal Sperner hypergraphs (that is, within the class of maximal clique hypergraphs of graphs), the classes of 1-Sperner hypergraphs, of threshold hypergraphs, and of 2-assummable hypergraphs coincide. These classes provide three new characterizations of threshold graphs: a graph is threshold if and only if its clique hypergraph is 1-Sperner, or equivalently threshold, or equivalently 2-assummable.

2 Results on 1-Sperner hypergraphs

Every hypergraph $\mathcal{H} = (V, \mathcal{E})$ with a fixed pair of orderings of its vertices and edges, say $V = \{v_1, \dots, v_n\}$, and $\mathcal{E} = \{e_1, \dots, e_m\}$, can be represented with its *incidence matrix* $A^{\mathcal{H}} \in \{0, 1\}^{m \times n}$ having rows and columns indexed by edges and vertices of \mathcal{H} , respectively, and being defined as

$$A_{i,j}^{\mathcal{H}} = \begin{cases} 1, & \text{if } v_j \in e_i; \\ 0, & \text{otherwise.} \end{cases}$$

Note the slight abuse of notation above: the incidence matrix does not depend only on the hypergraph but also on the pair of orderings of its vertices and edges. We will be able to neglect this technical issue often in the paper, but not always. We will therefore say that two matrices A and B of the same dimensions are *permutation equivalent*, and denote this fact by $A \cong B$, if A can be obtained from B by permuting some of its rows and/or columns. For later use, we state a simple property of incidence matrices of a hypergraph.

Remark 1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with a fixed pair of orderings of its vertices and edges, respectively, and let $A^{\mathcal{H}}$ be the corresponding incidence matrix. Then, any permutation of the vertices and/or edges of \mathcal{H} results in an incidence matrix that is permutation equivalent to $A^{\mathcal{H}}$. Moreover, any matrix that is permutation equivalent to $A^{\mathcal{H}}$ is the incidence matrix of \mathcal{H} with respect to some pair of orderings of its vertices and edges.*

We now describe an operation that produces a new 1-Sperner hypergraph from a given pair of 1-Sperner hypergraphs. Given a pair of vertex-disjoint hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ and a new vertex $z \notin V_1 \cup V_2$, the *gluing* of \mathcal{H}_1 and \mathcal{H}_2 is the hypergraph $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ such that

$$V(\mathcal{H}) = V_1 \cup V_2 \cup \{z\}$$

and

$$E(\mathcal{H}) = \{\{z\} \cup e \mid e \in \mathcal{E}_1\} \cup \{V_1 \cup e \mid e \in \mathcal{E}_2\}.$$

The operation of gluing can be visualized easily in terms of incidence matrices. Let $n_i = |V_i|$ and $m_i = |\mathcal{E}_i|$ for $i = 1, 2$, and let us denote by $\mathbf{0}^{k,\ell}$, resp. $\mathbf{1}^{k,\ell}$, the $k \times \ell$ matrix of all zeroes, resp. of all ones. Then, the incidence matrix of the gluing of \mathcal{H}_1 and \mathcal{H}_2 can be written as

$$A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{pmatrix} \mathbf{1}^{m_1,1} & A^{\mathcal{H}_1} & \mathbf{0}^{m_1,n_2} \\ \mathbf{0}^{m_2,1} & \mathbf{1}^{m_2,n_1} & A^{\mathcal{H}_2} \end{pmatrix}.$$

See Fig. 1 for an example.

$$\begin{array}{ccc}
A^{\mathcal{H}_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \longrightarrow & A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{array}{c|ccc|ccc} & & & & & & & \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 & \\ \hline & 1 & 0 & 1 & 0 & 0 & 0 & \\ \hline & 0 & 1 & 1 & 1 & 1 & 0 & \\ \hline & 0 & 1 & 1 & 1 & 0 & 1 & \\ \hline & 0 & 1 & 1 & 0 & 1 & 1 & \end{array} \\
A^{\mathcal{H}_2} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} & &
\end{array}$$

Figure 1: An example of gluing of two hypergraphs

Proposition 2.1. *For every pair $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ of vertex-disjoint 1-Sperner hypergraphs, their gluing $\mathcal{H}_1 \odot \mathcal{H}_2$ is a 1-Sperner hypergraph, unless $\mathcal{E}_1 = \{V_1\}$ and $\mathcal{E}_2 = \{\emptyset\}$ (in which case the hypergraph $\mathcal{H}_1 \odot \mathcal{H}_2$ is not Sperner).*

Proof. Let e and f be two distinct edges of $\mathcal{H}_1 \odot \mathcal{H}_2$. If $z \in e \cap f$ then their differences are the same as the corresponding differences of $e \setminus \{z\}$ and $f \setminus \{z\}$, both of which are hyperedges of \mathcal{H}_1 . If $z \notin e \cup f$ then their differences are the same as the corresponding differences of $e \setminus V_1$ and $f \setminus V_1$, both of which are hyperedges of \mathcal{H}_2 . If $z \in e \setminus f$, then $e \setminus f = \{z\}$ and $f \setminus e \neq \emptyset$, unless $e = V_1$, and $f = \emptyset$ (which implies $\mathcal{E}_1 = \{V_1\}$ and $\mathcal{E}_2 = \{\emptyset\}$ by our assumption that both \mathcal{H}_1 and \mathcal{H}_2 are 1-Sperner). The case of $z \in f \setminus e$ is symmetric. \square

The following unary operation also preserves the class of 1-Sperner hypergraphs. Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, the *complement* of \mathcal{H} is the hypergraph $\overline{\mathcal{H}}$ with $V(\overline{\mathcal{H}}) = V$ and $E(\overline{\mathcal{H}}) = \{\bar{e} \mid e \in E(\mathcal{H})\}$, where \bar{e} denotes $V \setminus e$ for any subset $e \subseteq V$.

Proposition 2.2. *The complement of every 1-Sperner hypergraph is 1-Sperner.*

Proof. This follows from the fact that for every two sets $e, f \subseteq V$, we have $e \setminus f = \bar{f} \setminus \bar{e}$ and $f \setminus e = \bar{e} \setminus \bar{f}$. \square

Given a vertex z of a hypergraph \mathcal{H} , we say that a hypergraph \mathcal{H} is *z -decomposable* if for every two hyperedges $e, f \in E(\mathcal{H})$ such that $z \in e \setminus f$, we have $e \setminus \{z\} \subseteq f$. Equivalently, if the vertex set of \mathcal{H} can be partitioned as $V(\mathcal{H}) = \{z\} \cup V_1 \cup V_2$ such that $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ for some hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$. We say that a hypergraph \mathcal{H} is *1-decomposable* if it is z -decomposable for some $z \in V(\mathcal{H})$.

A vertex u in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *universal*, resp., *isolated* if it is contained in all (resp., in no) hyperedges.

The following proposition gathers some basic properties of the gluing operation and decomposability.

Proposition 2.3. *Let \mathcal{H} be a hypergraph. Then, the following holds:*

- (i) *If $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$, then $\overline{\mathcal{H}} = \overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}$ (assuming that in both gluings the same new vertex is used).*
- (ii) *If z is a vertex of \mathcal{H} such that \mathcal{H} is z -decomposable, then $\overline{\mathcal{H}}$ is also z -decomposable.*
- (iii) *If u is an isolated or a universal vertex of \mathcal{H} , then \mathcal{H} is u -decomposable.*

Proof. Note that if \mathcal{K} and \mathcal{L} are two hypergraphs such that $\overline{\mathcal{K}} = \mathcal{L}$, then their incidence matrices are of the same dimensions, say $p \times q$, and $A^{\mathcal{K}} + A^{\mathcal{L}} = \mathbf{1}^{p,q}$ (in particular, we have $\overline{\overline{\mathcal{K}}} = \overline{\mathcal{L}} = \mathcal{K}$).

(i) Suppose that $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ and let z be the new vertex, which is also the new vertex in the gluing of $\overline{\mathcal{H}_2}$ and $\overline{\mathcal{H}_1}$. We prove the desired equality $\overline{\mathcal{H}} = \overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}$ by establishing permutation equivalence of the incidence matrices of \mathcal{H} and of $\mathcal{H}' = \overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}$ (cf. Remark 1). By convention, when representing the incidence matrix of the result of a gluing, we use the notation as in Fig. 1, that is, z indexes the first column, vertices of the first factor precede, as column indices, all vertices of the second factor, and hyperedges of the first factor precede, as row indices, all hyperedges of the second factor. Letting $n = |V(\mathcal{H})|$, $m = |E(\mathcal{H})|$, $n_i = |V(\mathcal{H}_i)|$ and $m_i = |E(\mathcal{H}_i)|$ for $i = 1, 2$, we thus have

$$\begin{aligned}
A^{\mathcal{H}'} &= A^{\overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}} \\
&= \mathbf{1}^{m,n} - A^{\overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}} \\
&= \mathbf{1}^{m,n} - \begin{pmatrix} \mathbf{1}^{m_2,1} & A^{\overline{\mathcal{H}_2}} & \mathbf{0}^{m_2,n_1} \\ \mathbf{0}^{m_1,1} & \mathbf{1}^{m_1,n_2} & A^{\overline{\mathcal{H}_1}} \end{pmatrix} \\
&= \mathbf{1}^{m,n} - \begin{pmatrix} \mathbf{1}^{m_2,1} & \mathbf{1}^{m_2,n_2} - A^{\mathcal{H}_2} & \mathbf{0}^{m_2,n_1} \\ \mathbf{0}^{m_1,1} & \mathbf{1}^{m_1,n_2} & \mathbf{1}^{m_1,n_2} - A^{\mathcal{H}_1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}^{m_2,1} & A^{\mathcal{H}_2} & \mathbf{1}^{m_2,n_1} \\ \mathbf{1}^{m_1,1} & \mathbf{0}^{m_1,n_2} & A^{\mathcal{H}_1} \end{pmatrix} \\
&\cong \begin{pmatrix} \mathbf{1}^{m_1,1} & A^{\mathcal{H}_1} & \mathbf{0}^{m_1,n_2} \\ \mathbf{0}^{m_2,1} & \mathbf{1}^{m_2,n_1} & A^{\mathcal{H}_2} \end{pmatrix} \\
&= A^{\mathcal{H}_1 \odot \mathcal{H}_2} \\
&= A^{\mathcal{H}}.
\end{aligned}$$

(ii) Let z be a vertex of \mathcal{H} such that \mathcal{H} is z -decomposable. Then, there exists a pair of vertex-disjoint hypergraphs \mathcal{H}_1 and \mathcal{H}_2 such that \mathcal{H} is the gluing of \mathcal{H}_1 and \mathcal{H}_2 with $V(\mathcal{H}) = V(\mathcal{H}_1) \cup V(\mathcal{H}_2) \cup \{z\}$. By the proof of (i) (and using the same notation), we have $A^{\overline{\mathcal{H}}} = A^{\overline{\mathcal{H}_2} \odot \overline{\mathcal{H}_1}} = \begin{pmatrix} \mathbf{1}^{m_2,1} & A^{\overline{\mathcal{H}_2}} & \mathbf{0}^{m_2,n_1} \\ \mathbf{0}^{m_1,1} & \mathbf{1}^{m_1,n_2} & A^{\overline{\mathcal{H}_1}} \end{pmatrix}$, where the first column is indexed by z . Thus, $\overline{\mathcal{H}}$ is z -decomposable.

(iii) Note that u is universal in \mathcal{H} if and only if it is isolated in $\overline{\mathcal{H}}$. By (ii), it therefore suffices to prove the statement for the case when u is an isolated vertex of \mathcal{H} . In this case, the column of $A^{\mathcal{H}}$ indexed by u is the all zero vector. It follows that \mathcal{H} is u -decomposable, as follows: $V(\mathcal{H}) = \{u\} \cup V_1 \cup V_2$ with $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$, $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V_1 = \mathcal{E}_1 = \emptyset$, $V_2 = V \setminus \{u\}$, and $\mathcal{E}_2 = E(\mathcal{H})$. \square

2.1 Two lemmas and two families of examples

Lemma 2.4. *Let C be a hyperedge of maximum possible size in a 1-Sperner hypergraph \mathcal{H} . Then, for every two distinct vertices $x, y \notin C$ and every two hyperedges A containing x and*

B containing y , $|A| \leq |B|$ implies $A \cap C \subseteq B \cap C$.

Proof. Note that $|A| \leq |C|$, therefore $A \setminus C = \{x\}$, since \mathcal{H} is 1-Sperner. Analogously, $B \setminus C = \{y\}$. Thus, if the sets $A \cap C$ and $B \cap C$ were not \subseteq -comparable, the pair $\{A, B\}$ would violate the 1-Sperner property of \mathcal{H} . \square

Given a positive integer r , a hypergraph \mathcal{H} is said to be r -uniform if $|e| = r$ for all $e \in E(\mathcal{H})$.

Lemma 2.5. *Let \mathcal{H} be an r -uniform 1-Sperner hypergraph, where $r \geq 1$. Then, either there is a subset P of vertices of size $r - 1$ such that $P \subseteq e$ for all $e \in E(\mathcal{H})$ or there is a subset Q of vertices of size $r + 1$ such that $e \subseteq Q$ for all $e \in E(\mathcal{H})$.*

Proof. The statement of the lemma holds if \mathcal{H} has at most one hyperedge. So let us assume that \mathcal{H} has at least two hyperedges, say e and f . Let $P = e \cap f$. Since \mathcal{H} is 1-Sperner, $|P| = r - 1$. If all hyperedges of \mathcal{H} contain P , then we are done.

If there is a hyperedge g such that $P \not\subseteq g$, say $u \in P \setminus g$, then e and f are the only hyperedges containing P , since otherwise g should contain all vertices of such hyperedges other than u , which would imply $|g| > r$. Consequently, all hyperedges that miss a vertex of P are subsets of $Q = e \cup f$, and the lemma is proved. \square

Lemma 2.5 suggests the following two families of uniform 1-Sperner hypergraphs.

Example 2.1. *Given $r \geq 1$, an r -star is an r -uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that there exists sets $X, Y \subseteq V$ such that*

- $X \cup Y \subseteq V$ where $|X| = r - 1$, $Y \neq \emptyset$, and $X \cap Y = \emptyset$, and
- $\mathcal{E} = \{X \cup \{y\} \mid y \in Y\}$.

If this is the case, we say that \mathcal{H} is the (r) -star generated by (V, X, Y) .

Clearly, every r -star is 1-Sperner. Moreover, let us verify that every r -star is z -decomposable with respect to every vertex z . Let \mathcal{H} be an r -star generated by (V, X, Y) and let $z \in V(\mathcal{H})$. If $z \in X$, then $r \geq 2$ and we have $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ where \mathcal{H}_1 is the $(r - 1)$ -star generated by $X \setminus \{z\}$ and Y and $V(\mathcal{H}_2) = E(\mathcal{H}_2) = \emptyset$. If $z \in Y$, then we have $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ where $\mathcal{H}_1 = (X, \{X\})$ and $\mathcal{H}_2 = (Y \setminus \{z\}, \{\{y\} \mid y \in Y \setminus \{z\}\})$. Finally, if $z \in V \setminus (X \cup Y)$, then z is isolated and \mathcal{H} is z -decomposable by Proposition 2.3.

Example 2.2. *Given $r \geq 1$, an r -antistar is an r -uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that there exists sets $X, Y \subseteq V$ such that*

- $X \cup Y \subseteq V$ where $Y \neq \emptyset$ and $X \cap Y = \emptyset$, and
- $\mathcal{E} = \{X \cup (Y \setminus \{y\}) \mid y \in Y\}$.

If this is the case, we say that \mathcal{H} is the (r) -antistar generated by (V, X, Y) . Note that every r -antistar is the complement of an r -star. It follows, using Propositions 2.2 and 2.3 and the properties of stars observed in Example 2.1, that every antistar is 1-Sperner and z -decomposable with respect to each vertex z .

2.2 A decomposition theorem

Theorem 2.6. *Every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $V \neq \emptyset$ is 1-decomposable, that is, it is the gluing of two 1-Sperner hypergraphs.*

Proof. By Proposition 2.3, we may assume that \mathcal{H} does not have any isolated vertices. For every $v \in V$, let

$$k(v) = \max_{e \in \mathcal{E}} |e|$$

and let $k = \max_{e \in \mathcal{E}} |e|$.

We consider two cases.

Case 1: Not all the $k(v)$ values are the same. Let $v \in V$ be a vertex with the smallest $k(v)$ value. Then $k(v) < k$ by the assumption of this case.

Suppose first that for every hyperedge $f \in \mathcal{E}$ such that $v \notin f$, we have $|f| \geq k(v)$. We claim that in this case \mathcal{H} is v -decomposable. This is because for every two hyperedges $e, f \in \mathcal{E}$ such that $v \in e \setminus f$, we have $|f| \geq k(v) \geq |e|$, implying $|f \setminus e| \geq |e \setminus f|$, from what we derive, using the fact that \mathcal{H} is 1-Sperner, that $|e \setminus f| = 1$, that is, $e \setminus \{v\} \subseteq f$. This proves the claim.

Assume next that there exists a hyperedge $f \in \mathcal{E}$ such that $v \notin f$ and $|f| < k(v)$. Let e be a hyperedge containing v of size $k(v)$, and let g be a hyperedge of maximum possible size, that is, $|g| = k$. Then $v \notin g$, since $k(v) < |g|$. Since \mathcal{H} is Sperner, there exists a vertex $u \in f \setminus g$. Note that $|f| \leq |g|$, therefore $f \setminus g = \{u\}$, since \mathcal{H} is 1-Sperner. Moreover, $u \neq v$ since $u \in f$ and f does not contain v .

We know that $k(u) \geq k(v)$, by our choice of v . Therefore, there exists a hyperedge h containing u and of size $k(u)$. Since $|h| = k(u) \geq k(v) > |f|$, we have $h \neq f$. Applying Lemma 2.4 with $(x, y, A, B, C) = (u, v, f, e, g)$ implies $f \cap g \subseteq e \cap g$. Applying Lemma 2.4 with $(x, y, A, B, C) = (v, u, e, h, g)$ implies $e \cap g \subseteq h \cap g$. Consequently, $f \cap g \subseteq h \cap g$. On the other hand, $f \setminus g = h \setminus g = \{u\}$. It follows that $f \subseteq h$, contradicting the Sperner property of \mathcal{H} . This completes Case 1.

Case 2: All the $k(v)$ values are the same. Let $v \in V(\mathcal{H})$ and let $k = k(v)$. If $k \leq 1$, then \mathcal{H} is z -decomposable with respect to every vertex z . So suppose that $k \geq 2$. Consider the subhypergraph \mathcal{H}' of \mathcal{H} with $V(\mathcal{H}') = V(\mathcal{H})$ formed by the hyperedges of \mathcal{H} of size k . By Lemma 2.5 applied to \mathcal{H}' , either there is a subset P of vertices of size $k - 1$ such that $P \subseteq e$ for all $e \in E(\mathcal{H}')$ or there is a subset Q of vertices of size $k + 1$ such that $e \subseteq Q$ for all $e \in E(\mathcal{H}')$.

Suppose first that there is a subset P of vertices of size $k - 1$ such that $P \subseteq e$ for all $e \in E(\mathcal{H}')$. If $\mathcal{H}' = \mathcal{H}$, that is, all hyperedges of \mathcal{H} are of size k , then \mathcal{H} is z -decomposable with respect to every vertex z (cf. Example 2.1). So we may assume that $\mathcal{H}' \neq \mathcal{H}$, that is, that \mathcal{H} contains a hyperedge g of size less than k . By the assumption of Case 2, we know that $g \subseteq \bigcup_{f \in E(\mathcal{H}')} f$. Since \mathcal{H} is Sperner, g is not contained in any of the hyperedges of \mathcal{H}' ; moreover g contains at least two vertices from the set $Y = (\bigcup_{f \in E(\mathcal{H}')} f) \setminus P$. If g contains at least three vertices from Y , say y_1, y_2, y_3 , then the sets $P \cup \{y_1\}$ and g would violate the 1-Sperner property, since $\{y_2, y_3\} \subseteq g \setminus (P \cup \{y_1\})$ and $P \setminus g \subseteq (P \cup \{y_1\}) \setminus g$ (note that

$|P \setminus g| \geq 3$). It follows that $|g \cap Y| = 2$. In fact, we have $|Y| = 2$, say $Y = \{y_1, y_2\}$, since otherwise, using similar arguments as above, we see that the sets $P \cup \{y\}$ and g would violate the 1-Sperner property, where $y \in Y \setminus g$. It follows that \mathcal{H}' has exactly 2 hyperedges, and $Y \subseteq e$ for every set $e \in E(\mathcal{H}) \setminus E(\mathcal{H}')$. Consequently, \mathcal{H} is y -decomposable for every $y \in Y$: Decomposing \mathcal{H} with respect to $y = y_1$, for instance, we have $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ where $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ with $V_1 = V \setminus \{y_1\}$, $\mathcal{E}_1 = \{e \setminus \{y_1\} \mid y_1 \in e \in E(\mathcal{H})\}$, and $\mathcal{H}_2 = (\emptyset, \{\emptyset\})$.

It remains to consider the case when there is a subset Q of vertices of size $k + 1$ such that $e \subseteq Q$ for all $e \in E(\mathcal{H}')$. Since we assume that $k(v) = k$ for all vertices v , we have $V = Q$. Let us define $X = \bigcap_{f \in E(\mathcal{H}')} f$ and $Y = V \setminus X$. Then, $Y \neq \emptyset$, and every hyperedge $g \in E(\mathcal{H}) \setminus E(\mathcal{H}')$ must contain Y , since \mathcal{H} is Sperner and for every vertex $y \in Y$, the set $V \setminus \{y\}$ is a hyperedge of \mathcal{H} . Consequently, \mathcal{H} is y -decomposable for every $y \in Y$: taking any $y \in Y$, we have $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ where $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ with $V_1 = V \setminus \{y\}$, $\mathcal{E}_1 = \{e \setminus \{y\} \mid y \in e \in E(\mathcal{H})\}$, and $\mathcal{H}_2 = (\emptyset, \{\emptyset\})$. \square

Let us say that a gluing of two vertex-disjoint hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ is *safe* if it results in a 1-Sperner hypergraph. By Proposition 2.1, this is equivalent to requiring that either $\mathcal{E}_1 \neq \{V_1\}$ or $\mathcal{E}_2 \neq \{\emptyset\}$. Theorem 2.6 and Proposition 2.1 imply the following composition result for the class of 1-Sperner hypergraphs.

Theorem 2.7. *A hypergraph \mathcal{H} is 1-Sperner if and only if it either has no vertices (that is, $\mathcal{H} \in \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\})\}$) or it is a safe gluing of two smaller 1-Sperner hypergraphs.*

2.3 Consequences of Theorem 2.7

Here, we give a new proof of the thresholdness property of 1-Sperner hypergraphs. Recall that it was shown in [5] that every dually Sperner hypergraph is threshold; in particular, every 1-Sperner hypergraph is threshold. That proof is based on the characterization of thresholdness in terms of k -asummability (see Section 3). The proof given here is based on the composition theorem for 1-Sperner hypergraphs (Theorem 2.7) and, unlike the proof from [5], also constructs the weights and a threshold as in the definition of threshold hypergraphs.

Proposition 2.8 (Chiarelli-Milanić [5]). *Every 1-Sperner hypergraph is threshold.*

Proof. We will show by induction on $n = |V|$ that for every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ there exists a *nice threshold separator*, that is, a pair (w, t) such that $w : V \rightarrow \mathbb{Z}_{>0}$ is a strictly positive integer weight function and $t \in \mathbb{Z}_{\geq 0}$ is a non-negative integer threshold such that:

- (i) for every subset $X \subseteq V$, we have $w(X) \geq t$ if and only if $e \subseteq X$ for some $e \in \mathcal{E}$;
- (ii) if $w(V) = t$ then $\mathcal{E} = \{V\}$, and
- (iii) if $t = 0$ then $\mathcal{E} = \{\emptyset\}$.

In particular, this will establish the thresholdness of \mathcal{H} .

For $n = 0$, we can take the (empty) mapping given by $w(x) = 1$ for all $x \in V$ and the threshold

$$t = \begin{cases} 1, & \text{if } \mathcal{E} = \emptyset; \\ 0, & \text{if } \mathcal{E} = \{\emptyset\}. \end{cases}$$

Now, let $\mathcal{H} = (V, \mathcal{E})$ be a 1-Sperner hypergraph with $n \geq 1$. By Theorem 2.7, \mathcal{H} is the safe gluing of two 1-Sperner hypergraphs, say $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V = V_1 \cup V_2 \cup \{z\}$, $V_1 \cap V_2 = \emptyset$, and $z \notin V_1 \cup V_2$. By the inductive hypothesis, \mathcal{H}_1 and \mathcal{H}_2 admit nice threshold separators. That is, there exist positive integer weight functions $w_i : V_i \rightarrow \mathbb{Z}_{>0}$ and non-negative integer thresholds $t_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2$ such that for every subset $X \subseteq V_i$, we have $w_i(X) \geq t_i$ if and only if $e \subseteq X$ for some $e \in \mathcal{E}_i$. Moreover, if $w_i(V_i) = t_i$ then $\mathcal{E}_i = \emptyset$, and if $t_i = 0$ then $\mathcal{E}_i = \{\emptyset\}$.

Let us define the threshold $t = Mw_1(V_1) + t_2$, where $M = w_2(V_2) + 1$, and the weight function $w : V \rightarrow \mathbb{Z}_{>0}$ by the rule

$$w(x) = \begin{cases} Mw_1(x), & \text{if } x \in V_1; \\ w_2(x), & \text{if } x \in V_2; \\ M(w_1(V_1) - t_1) + t_2, & \text{if } x = z. \end{cases}$$

Let us verify that the so defined weight function is indeed strictly positive. Since w_i for $i \in \{1, 2\}$ are strictly positive and $M > 0$, we have $w(x) > 0$ for all $x \in V_1 \cup V_2$. Moreover, since $w_1(V_1) \geq t_1$, $M \geq 0$, and $t_2 \geq 0$, we have $w(z) \geq 0$. If $w(z) = 0$, then $w_1(V_1) = t_1$ and $t_2 = 0$, which implies $\mathcal{E}_1 = \{V_1\}$ and $\mathcal{E}_2 = \{\emptyset\}$, contrary to the fact that the gluing is safe. It follows that $w(z) > 0$, as claimed.

We claim that (w, t) is a nice threshold separator of \mathcal{H} . We first verify property (ii). Suppose that $w(V) = t$. Then $Mw_1(V_1) + w_2(V_2) + w(z) = t$, which is equivalent to $Mw_1(V_1) + w_2(V_2) = Mt_1$. This implies $w_1(V_1) \leq t_1$. If $w_1(V_1) \leq t_1 - 1$, then $Mw_1(V_1) + w_2(V_2) \leq Mt_1 - M + w_2(V_2) < Mt_1$, a contradiction. Therefore, $w_1(V_1) = t_1$, which by the inductive hypothesis implies $\mathcal{E}_1 = \{V_1\}$. Moreover, we have $w_2(V_2) = 0$, which implies $V_2 = \emptyset$. Consequently, $\mathcal{E} = \{\{z\} \cup V_1\} = \{V\}$, as claimed.

Now, we verify property (iii). Suppose that $t = 0$. This implies $w_1(V_1) = 0$ and $t_2 = 0$, and consequently $V_1 = \emptyset$ (since w_1 is strictly positive) and $\mathcal{E}_2 = \{\emptyset\}$ (by the inductive hypothesis). Since the gluing is safe and $\mathcal{E}_2 = \{\emptyset\}$, it follows that $\mathcal{E}_1 \neq \{V_1\}$, hence, using $V_1 = \emptyset$, we infer $\mathcal{E}_1 = \emptyset$. We thus have $\mathcal{E} = \{\{z\} \cup e \mid e \in \mathcal{E}_1\} \cup \{V_1 \cup e \mid e \in \mathcal{E}_2\} = \{\emptyset\}$, as claimed.

Finally, we verify property (i), that is, that for every subset $X \subseteq V$, we have $w(X) \geq t$ if and only if $e \subseteq X$ for some $e \in \mathcal{E}$.

Suppose first that $w(X) \geq t$ for some $X \subseteq V$. Let $X_i = X \cap V_i$ for $i = 1, 2$. For later use, we note that

$$w_2(X_2) \leq w_2(V_2) < M. \quad (3)$$

Suppose first that $z \in X$. Then

$$\begin{aligned} Mw_1(V_1) + t_2 &= t \leq w(X) \\ &= w(z) + w(X_1) + w(X_2) \\ &= M(w_1(V_1) - t_1) + t_2 + Mw_1(X_1) + w_2(X_2), \end{aligned}$$

which implies

$$Mw_1(X_1) + w_2(X_2) \geq Mt_1. \quad (4)$$

If $w_1(X_1) \leq t_1 - 1$ then, using (3), we obtain

$$Mw_1(X_1) + w_2(X_2) \leq Mt_1 - M + w_2(X_2) < Mt_1,$$

a contradiction with (4). It follows that $w_1(X_1) \geq t_1$. Consequently there exists $e_1 \in \mathcal{E}_1$ such that $e_1 \subseteq X_1$, hence the hyperedge $e := \{z\} \cup e_1 \in \mathcal{E}$ satisfies $e \subseteq X$.

Now, suppose that $z \notin X$. In this case,

$$Mw_1(V_1) + t_2 = t \leq w(X) = w(X_1) + w(X_2) = Mw_1(X_1) + w_2(X_2),$$

which implies

$$Mw_1(X_1) + w_2(X_2) \geq Mw_1(V_1) + t_2. \quad (5)$$

We must have $X_1 = V_1$ since if there exists a vertex $v \in V_1 \setminus X_1$, then we would have

$$\begin{aligned} Mw_1(X_1) + w_2(X_2) &\leq Mw_1(V_1) - Mw_1(v) + w_2(X_2) \\ &\leq Mw_1(V_1) - M + w_2(X_2) \\ &< Mw_1(V_1) + t_2, \end{aligned}$$

where the last inequality follows from (3) and $t_2 \geq 0$. Therefore, inequality (8) simplifies to $w_2(X_2) \geq t_2$, and consequently there exists a hyperedge $e_2 \in \mathcal{E}_2$ such that $e_2 \subseteq X_2$. This implies that \mathcal{H} has a hyperedge $e := V_1 \cup e_2$ such that $e \subseteq V_1 \cup X_2 = X$.

For the converse direction, suppose that X is a subset of V such that $e \subseteq X$ for some $e \in \mathcal{E}$. We need to show that $w(X) \geq t$. We consider two cases depending on whether $z \in e$ or not. Suppose first that $z \in e$. Then $e = \{z\} \cup e_1$ for some $e_1 \in \mathcal{E}_1$. Due to the property of w_1 , we have $w_1(e_1) \geq t_1$. Consequently,

$$\begin{aligned} w(X) &\geq w(e) \\ &= w(z) + w(e_1) \\ &= M(w_1(V_1) - t_1) + t_2 + Mw_1(e_1) \\ &\geq Mw_1(V_1) - Mt_1 + t_2 + Mt_1 \\ &= Mw_1(V_1) + t_2 = t. \end{aligned}$$

Suppose now that $z \notin e$. Then $e = V_1 \cup e_2$ for some $e_2 \in \mathcal{E}_2$. Due to the property of w_2 , we have $w_2(e_2) \geq t_2$. Consequently,

$$\begin{aligned} w(X) &= w(V_1) + w(e_2) \\ &= Mw_1(V_1) + w_2(e_2) \\ &\geq Mw_1(V_1) + t_2 = t. \end{aligned}$$

This shows that $w(X) \geq t$ whenever X contains a hyperedge of \mathcal{H} , and completes the proof. \square

The same inductive construction of weights and essentially the same proof shows that every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ is also *equilizable*, which means that there exist a non-negative integer weight function $w : V \rightarrow \mathbb{Z}_{\geq 0}$ and a non-negative integer threshold $t \in \mathbb{Z}_{\geq 0}$ such that for every subset $X \subseteq V$, we have $w(X) = t$ if and only if $X \in \mathcal{E}$. Note that equilizable hypergraphs generalize the class of equistable graphs studied, e.g., in [1, 3, 11–13, 15, 16, 18], in that a graph G is equistable if and only if its maximal stable set hypergraph is equilizable. The *maximal stable set hypergraph* of a graph G is the hypergraph $\mathcal{S}(G)$ with vertex set $V(G)$ and in which the hyperedges are exactly the maximal stable sets of G .

Proposition 2.9. *Every 1-Sperner hypergraph is equilizable.*

For the sake of completeness, we include the proof of Proposition 2.9 in Appendix.

By $\mathbf{0}$, resp. $\mathbf{1}$, we will denote the vector of all zeroes, resp. ones, of appropriate dimension (which will be clear from the context). The following lemma can be easily derived from Proposition 2.9.

Lemma 2.10. *For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\mathcal{E} \neq \emptyset$ and $\mathcal{E} \neq \{\emptyset\}$, there exists a vector $x \in \mathbb{R}_+^V$ such that $A^{\mathcal{H}}x = \mathbf{1}$ and $\mathbf{1}^\top x \geq 1$.*

Proof. Let $\mathcal{H} = (V, \mathcal{E})$ be a 1-Sperner hypergraph as in the statement of the lemma. By Proposition 2.9, \mathcal{H} is equilizable. Let $w : V \rightarrow \mathbb{Z}_{\geq 0}$ be a non-negative integer weight function and $t \in \mathbb{Z}_{\geq 0}$ a non-negative integer threshold such that for every subset $X \subseteq V$, we have $w(X) = t$ if and only if $X \in \mathcal{E}$. If $t = 0$, then $\emptyset \in \mathcal{E}$ and consequently $\mathcal{E} = \{\emptyset\}$, a contradiction. It follows that $t > 0$, and we can define the vector $x \in \mathbb{R}_+^V$ given by $x_v = w(v)/t$ for all $v \in V$. We claim that vector x satisfies the desired properties $A^{\mathcal{H}}x = \mathbf{1}$ and $\mathbf{1}^\top x \geq 1$.

Since $w(X) = t$ for all $X \in \mathcal{E}$, we have $A^{\mathcal{H}}x = \mathbf{1}$. Since $\mathcal{E} \neq \emptyset$, an arbitrary hyperedge $e \in \mathcal{E}$ shows that $t = w(e) \leq w(V)$. Consequently, we also have $\mathbf{1}^\top x = \sum_{v \in V} x_v = w(V)/t \geq 1$. \square

Corollary 2.11. *For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ and every vector $\lambda \in \mathbb{R}^{\mathcal{E}}$ we have*

$$\lambda^\top A^{\mathcal{H}} = \mathbf{1}^\top \Rightarrow \lambda^\top \mathbf{1} \geq 1.$$

Proof. If $\mathcal{E} = \emptyset$ or $\mathcal{E} = \{\emptyset\}$, then the left hand side of the above implication is always false. In all other cases, by Lemma 2.10, there exists a vector $x \in \mathbb{R}^V$ such that $A^{\mathcal{H}}x = \mathbf{1}$ and $\mathbf{1}^\top x \geq 1$. Therefore, equation $\lambda^\top A^{\mathcal{H}} = \mathbf{1}^\top$ implies

$$\lambda^\top \mathbf{1} = \lambda^\top A^{\mathcal{H}}x = \mathbf{1}^\top x \geq 1.$$

\square

Proposition 2.12. *For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\mathcal{E} \neq \{\emptyset\}$, the characteristic vectors of its hyperedges are linearly independent in \mathbb{R}^V .*

Proof. We use induction on $|V|$. If $|V| \leq 1$, then the statement holds since $\mathcal{E} \neq \{\emptyset\}$.

Suppose now that $|V| > 1$. Then by Theorem 2.6, \mathcal{H} is the gluing of two 1-Sperner hypergraphs, say $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V = V_1 \cup V_2 \cup \{z\}$, $V_1 \cap V_2 = \emptyset$, and $z \notin V_1 \cup V_2$.

Let $\lambda \in \mathbb{R}^{\mathcal{E}}$ be a vector such that $\lambda^\top A^{\mathcal{H}} = \mathbf{0}$. Let λ^1 and λ^2 be the restrictions of λ to the hyperedges corresponding to \mathcal{E}_1 and \mathcal{E}_2 , respectively. The equation $\lambda^\top A^{\mathcal{H}} = \mathbf{0}$ implies the system of equations

$$\begin{aligned} (\lambda^1)^\top \mathbf{1} &= 0 \in \mathbb{R}, \\ (\lambda^1)^\top A^{\mathcal{H}_1} + ((\lambda^2)^\top \mathbf{1}) \mathbf{1}^\top &= \mathbf{0}^\top \in \mathbb{R}^{V_1}, \\ (\lambda^2)^\top A^{\mathcal{H}_2} &= \mathbf{0}^\top \in \mathbb{R}^{V_2}. \end{aligned}$$

In all cases, the inductive hypothesis implies that $\lambda^1 = \mathbf{0}^\top \in \mathbb{R}^{\mathcal{E}_1}$ and $\lambda^2 = \mathbf{0}^\top \in \mathbb{R}^{\mathcal{E}_2}$, except in the case when $\mathcal{E}_2 = \{\emptyset\}$. In this case, λ_2 is a single number, say λ^* . If $\lambda^* = 0$, then $\lambda_1 = \mathbf{0}^\top$ follows by the induction hypothesis. If $\lambda^* \neq 0$, then $\hat{\lambda} := -\lambda_1/\lambda^*$ satisfies $\hat{\lambda}^\top A^{\mathcal{H}_1} = \mathbf{1}^\top$ and $\hat{\lambda}^\top \mathbf{1} = \mathbf{0}$, contradicting Corollary 2.11. \square

Corollary 2.13. *For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $V \neq \emptyset$, we have $|\mathcal{E}| \leq |V|$.*

Recall that a vertex u in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *universal*, resp., *isolated* if it is contained in all (resp., in no) hyperedges. Moreover, two vertices u, v in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ are *twins* if they are contained in exactly the same hyperedges.

Corollary 2.13 gives an upper bound on the number of hyperedges of a 1-Sperner hypergraph in terms of the number of vertices. Can we prove a lower bound of a similar form? In general not, since adding universal vertices, isolated vertices, or twin vertices preserves the 1-Sperner property and the number of edges, while it increases the number of vertices. However, as we show next, for 1-Sperner hypergraphs without universal, isolated vertices, and twin vertices, we prove the following sharp lower bound on the number of hyperedges.

Proposition 2.14. *For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| \geq 2$ and without universal, isolated, and twin vertices, we have*

$$|\mathcal{E}| \geq \left\lceil \frac{|V| + 2}{2} \right\rceil.$$

This bound is sharp.

Proof. We use induction on $n = |V|$. For $n \in \{2, 3, 4\}$, it can be easily verified that the statement holds.

Now, let $\mathcal{H} = (V, \mathcal{E})$ be a 1-Sperner hypergraph with $n \geq 5$ and without universal vertices, isolated vertices, and twin vertices. By Theorem 2.6, \mathcal{H} is the gluing of two 1-Sperner hypergraphs, say $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V = V_1 \cup V_2 \cup \{z\}$, $V_1 \cap V_2 = \emptyset$, and $z \notin V_1 \cup V_2$. Let $n_i = |V_i|$ and $m_i = |\mathcal{E}_i|$ for $i = 1, 2$, and let $m = |\mathcal{E}|$.

We have $m = m_1 + m_2$, and by the rules of the gluing, $n = n_1 + n_2 + 1$. By Proposition 2.2, we may assume that $n_1 \geq n_2$ (otherwise, we can consider the complementary hypergraph). In particular, $n_1 \geq 3$. The fact that \mathcal{H} does not have a universal vertex implies \mathcal{H}_1 does not have a universal vertex. Similarly, \mathcal{H}_2 does not have an isolated vertex. Since \mathcal{H} does not have any pairs of twin vertices, we have that either \mathcal{H}_1 does not have a isolated vertex, or \mathcal{H}_2 does not have a universal vertex. We may assume that \mathcal{H}_2 does not have a universal vertex (otherwise, we consider a different gluing in which we delete the universal vertex from \mathcal{H}_2 and add an isolated vertex to \mathcal{H}_1).

First, suppose that $n_2 \geq 2$, then we apply the inductive hypothesis for \mathcal{H}'_1 and \mathcal{H}_2 , where \mathcal{H}'_1 is the hypergraph obtained from \mathcal{H}_1 by deleting from it the isolated vertex (if it exists). Letting $n'_1 = |V(\mathcal{H}'_1)|$ and $m'_1 = |E(\mathcal{H}'_1)|$, we thus have $n'_1 \geq n_1 - 1$ and also $n'_1 \geq 2$. We obtain

$$m_1 = m'_1 \geq \frac{n'_1 + 2}{2} \geq \frac{n_1 + 1}{2}$$

and

$$m_2 \geq \frac{n_2 + 2}{2}.$$

Consequently,

$$m = m_1 + m_2 \geq \frac{n_1 + 1}{2} + \frac{n_2 + 2}{2} = \frac{n_1 + n_2 + 3}{2} = \frac{n + 2}{2},$$

and, since m is integer, the desired inequality

$$m \geq \left\lceil \frac{n + 2}{2} \right\rceil$$

follows.

Suppose now that $n_2 = 1$. In this case we have $m_2 = 1$, by the Sperner property of \mathcal{H}_2 . Moreover, \mathcal{H}_1 does not have an isolated vertex, since such a vertex would be a twin in \mathcal{H} of the unique vertex of \mathcal{H}_2 . Thus, we can apply the inductive hypothesis on \mathcal{H}_1 , which yields

$$m_1 \geq \frac{n_1 + 2}{2}$$

and consequently

$$m = m_1 + 1 \geq \frac{n_1 + 2}{2} + 1 = \frac{n_1 + 4}{2} = \frac{n + 2}{2},$$

as desired.

Finally, suppose that $n_2 = 0$. In this case, since \mathcal{H} does not have a universal vertex, we must have $\mathcal{E}_2 = \{\emptyset\}$ and $m_2 = 1$. As above, let \mathcal{H}'_1 be the hypergraph obtained from \mathcal{H}_1 by deleting from it the isolated vertex (if it exists). Letting $n'_1 = |V(\mathcal{H}'_1)|$ and $m'_1 = |E(\mathcal{H}'_1)|$, we obtain, by applying the inductive hypothesis to \mathcal{H}'_1 ,

$$m_1 = m'_1 \geq \frac{n'_1 + 2}{2} \geq \frac{n_1 + 1}{2},$$

which implies

$$m = m_1 + 1 \geq \frac{n_1 + 1}{2} + 1 = \frac{n}{2} + 1 = \frac{n + 2}{2}.$$

This completes the proof of the inequality.

To see that the inequality is sharp, consider the following family of hypergraphs. For $k \geq 2$, let \mathcal{H}_k be the hypergraph defined recursively as follows:

- $\mathcal{H}_2 = (\{v_1, v_2\}, \{\{v_1\}, \{v_2\}\})$.
- For $k > 2$, we set $\mathcal{H}_k = \mathcal{H}'_{k-1} \odot \mathcal{H}_{k-1}$ where \mathcal{H}'_{k-1} is the hypergraph obtained from a disjoint copy of \mathcal{H}_{k-1} by adding to it an isolated vertex.

An inductive argument shows that for every $k \geq 2$, we have $n_k = |V(\mathcal{H}_k)| = 2^k - 2$, $m_k = |E(\mathcal{H}_k)| = 2^{k-1}$, and consequently $m_k = \lceil \frac{n_k + 2}{2} \rceil$. \square

3 New characterizations of threshold graphs

A *threshold graph* is a threshold hypergraph in which all hyperedges are of size 2. Threshold graphs were introduced by Chvátal and Hammer in 1970s [7] and were afterwards studied in numerous paper. Many results on threshold graphs are summarized in the monograph by Mahadev and Peled [14].

Threshold graphs have many different characterizations. We now recall two of them and then prove three more. For a set \mathcal{F} of graphs, we say that a graph G is \mathcal{F} -free if no induced subgraph of G is isomorphic to a member of \mathcal{F} . By P_4 we denote the 4-vertex path, by C_4 the 4-vertex cycle, and by $2K_2$ the disjoint union of two copies of K_2 (the complete graph of order two).

Theorem 3.1 (Chvátal and Hammer [7]). *A graph G is threshold if and only if G is $\{P_4, C_4, 2K_2\}$ -free.*

Theorem 3.2 (Chvátal and Hammer [7]). *A graph G is threshold if and only if $V(G) = K \cup I$ where K is a clique, I is an independent set, $K \cap I = \emptyset$, and there exists an ordering v_1, \dots, v_k of I such that $N(v_i) \subseteq N(v_j)$ for all $1 \leq i < j \leq k$.*

Given a graph G , the *clique hypergraph* of G is the hypergraph $\mathcal{C}(G)$ with vertex set $V(G)$ in which the hyperedges are exactly the maximal cliques of G . The clique hypergraphs of graphs are exactly those Sperner hypergraphs \mathcal{H} that are also *normal* (or: *conformal*), that is, for every set $X \subseteq V(\mathcal{H})$ such that every pair of elements in X is contained in a hyperedge, there exists a hyperedge containing X (see [2].)

A necessary condition for thresholdness of a hypergraph \mathcal{H} is k -asummability for any $k \geq 2$. A hypergraph is k -asummable if it has no k (not necessarily distinct) independent sets A_1, \dots, A_k and k (not necessarily distinct) dependent sets B_1, \dots, B_k such that

$$\sum_{i=1}^k \chi^{A_i} = \sum_{i=1}^k \chi^{B_i}.$$

A hypergraph is known to be threshold if and only if it is k -asummable for all k [6, 9]. (More recent and complete information on this can be found in [8].) It is also known that 2-asummability does not imply thresholdness in general [8], and thresholdness does not imply 1-Spernerness (as can be seen by considering the edge set of the complete graph K_4).

As the next theorem shows, in the class of conformal Sperner hypergraphs, the notion of thresholdness coincides with both 2-asummability and 1-Spernerness. Moreover, it exactly characterizes threshold graphs.

Theorem 3.3. *For a graph G , the following statements are equivalent:*

- (1) G is threshold.
- (2) The clique hypergraph $\mathcal{C}(G)$ is 1-Sperner.
- (3) The clique hypergraph $\mathcal{C}(G)$ is threshold.
- (4) The clique hypergraph $\mathcal{C}(G)$ is 2-asummable.

Proof. The implication (1) \Rightarrow (2) follows directly from Theorem 3.2.

The implication (2) \Rightarrow (3) is the statement of Proposition 2.8.

The implication (3) \Rightarrow (4) follows from [6, 9].

Now we prove the implication (4) \Rightarrow (1). We break the proof into a small series of simple claims.

Claim 1. *For any graph G , if $A, B \in \mathcal{C}(G)$, $a \in A \setminus B$, and $b \in B \setminus A$ such that $(a, b) \in E(G)$, then the set $(A \setminus \{a\}) \cup \{b\}$ is an independent set of the hypergraph $\mathcal{C}(G)$.*

Proof. If $C \subseteq (A \setminus \{a\}) \cup \{b\}$ is a clique of G , then $C \cup \{a\}$ is also a clique of G , and hence C is not maximal. \square

Claim 2. *If $\mathcal{C}(G)$ is 2-asummable, and $A, B \in \mathcal{C}(G)$, $a \in A \setminus B$, and $b \in B \setminus A$ then $(a, b) \notin E(G)$.*

Proof. If $(a, b) \in E(G)$ then Claim 1 and the equality

$$\chi^A + \chi^B = \chi^{(A \setminus \{a\}) \cup \{b\}} + \chi^{(B \setminus \{b\}) \cup \{a\}}$$

contradicts the 2-asummability of $\mathcal{C}(G)$. \square

Claim 3. *If $\mathcal{C}(G)$ is 2-asummable, and $A, B \in \mathcal{C}(G)$, $a \in A \setminus B$, and $b \in B \setminus A$, then the set $(A \cup B) \setminus \{a, b\}$ is an independent set of the hypergraph $\mathcal{C}(G)$.*

Proof. If $C \subseteq (A \cup B) \setminus \{a, b\}$ is a maximal clique of G , then

- $C \cup \{a\}$ is not a clique, and hence there exists a vertex $u \in (B \cap C) \setminus A$ (such that $\{a, u\} \notin E(G)$);
- $C \cup \{b\}$ is not a clique, and hence there exists a vertex $v \in (A \cap C) \setminus B$ (such that $\{b, v\} \notin E(G)$).

Consequently we have vertices $u \in B \setminus A$ and $v \in A \setminus B$ such that $\{u, v\} \subseteq C$ implying $\{u, v\} \in E(G)$. This contradicts Claim 2, which proves that such a maximal clique C cannot exist. \square

Claim 4. *If $\mathcal{C}(G)$ is 2-asummable then G is a threshold graph.*

Proof. Let us recall the forbidden subgraph characterization of threshold graphs by Theorem 3.1, and assume indirectly that there exists four distinct vertices $\{a, b, c, d\} \subseteq V(G)$ such that $\{a, b\}, \{c, d\} \in E(G)$ and $\{a, c\}, \{b, d\} \notin E(G)$ (that is, that $\{a, b, c, d\}$ induces either a P_4 , a C_4 , or a $2K_2$.) Let us then consider maximal cliques $A \supseteq \{a, b\}$ and $B \supseteq \{c, d\}$ of G . Since $\{a, b\}, \{c, d\} \in E(G)$ such maximal cliques do exist.

Our assumptions that $\{a, c\}, \{b, d\} \notin E(G)$ imply that $\{a, b\} \subseteq A \setminus B$ and $\{c, d\} \subseteq B \setminus A$.

Then we can apply Claim 3 and obtain that both $(A \cup B) \setminus \{a, c\}$ and $(A \cup B) \setminus \{b, d\}$ are independent sets of the hypergraph $\mathcal{C}(G)$. Since $(A \cap B) \cup \{a, c\} \subseteq (A \cup B) \setminus \{b, d\}$, it also follows that $(A \cap B) \cup \{a, c\}$ is an independent set of $\mathcal{C}(G)$. Consequently, the equality

$$\chi^A + \chi^B = \chi^{(A \cup B) \setminus \{a, c\}} + \chi^{(A \cap B) \cup \{a, c\}}$$

contradicts the 2-asummability of $\mathcal{C}(G)$. This contradiction proves that such a set of four vertices cannot exist, from which the claim follows by Theorem 3.1. \square

The last claim proves the implication (4) \Rightarrow (1), completing the proof of the theorem. \square

Finally, we remark that, since the class of threshold graphs is closed under taking complements, one could obtain three further characterizations of threshold graphs by replacing the clique hypergraph $\mathcal{C}(G)$ with the maximal stable set hypergraph $\mathcal{S}(G)$ in any of the properties (2)–(4) in Theorem 3.3.

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Appendix: Proof of Proposition 2.9

Proposition 2.9. *Every 1-Sperner hypergraph is equilizable.*

Proof. We will show by induction on $n = |V|$ that for every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ there exists a strictly positive integer weight function $w : V \rightarrow \mathbb{Z}_{>0}$ and a non-negative integer threshold $t \in \mathbb{Z}_{\geq 0}$ such that for every subset $X \subseteq V$, we have $w(X) = t$ if and only if $X \in \mathcal{E}$. This will establish the equilizability of \mathcal{H} .

For $n = 0$, we can take the (empty) mapping given by $w(x) = 1$ for all $x \in V$ and the threshold

$$t = \begin{cases} 1, & \text{if } \mathcal{E} = \emptyset; \\ 0, & \text{if } \mathcal{E} = \{\emptyset\}. \end{cases}$$

Now, let $\mathcal{H} = (V, \mathcal{E})$ be a 1-Sperner hypergraph with $n \geq 1$. By Theorem 2.7, \mathcal{H} is a safe gluing of two 1-Sperner hypergraphs, say $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V = V_1 \cup V_2 \cup \{z\}$, $V_1 \cap V_2 = \emptyset$, and $z \notin V_1 \cup V_2$. By the inductive hypothesis, \mathcal{H}_1 and \mathcal{H}_2 are equilizable, that is, there exist positive integer weight functions $w_i : V_i \rightarrow \mathbb{Z}_{>0}$ and non-negative integer thresholds $t_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2$ such that for every subset $X \subseteq V_i$, we have $w_i(X) = t_i$ if and only if $X \in \mathcal{E}_i$.

Let us define the threshold $t = Mw_1(V_1) + t_2$, where $M = w_2(V_2) + 1$, and the weight function $w : V \rightarrow \mathbb{Z}_{>0}$ by the rule

$$w(x) = \begin{cases} Mw_1(x), & \text{if } x \in V_1; \\ w_2(x), & \text{if } x \in V_2; \\ M(w_1(V_1) - t_1) + t_2, & \text{if } x = z. \end{cases}$$

Since the weight function defined above coincides with the weight function defined in the proof of Proposition 2.8, the function is indeed strictly positive.

We claim that for every subset $X \subseteq V$, we have $w(X) = t$ if and only if $X \in \mathcal{E}$. This will establish the equilizability of \mathcal{H} .

Suppose first that $w(X) = t$ for some $X \subseteq V$. Let $X_i = X \cap V_i$ for $i = 1, 2$. For later use, we note that

$$w_2(X_2) \leq w_2(V_2) < M. \quad (6)$$

We consider two cases depending on whether $z \in X$ or not. Suppose first that $z \in X$. Then

$$\begin{aligned} Mw_1(V_1) + t_2 &= t = w(X) \\ &= w(z) + w(X_1) + w(X_2) \\ &= M(w_1(V_1) - t_1) + t_2 + Mw_1(X_1) + w_2(X_2), \end{aligned}$$

which implies

$$Mw_1(X_1) + w_2(X_2) = Mt_1. \quad (7)$$

If $w_1(X_1) \leq t_1 - 1$ then, using (6), we obtain

$$Mw_1(X_1) + w_2(X_2) \leq Mt_1 - M + w_2(X_2) < Mt_1,$$

a contradiction with (7). Therefore $w_1(X_1) \geq t_1$. Moreover, if $w_1(X_1) \geq t_1 + 1$, then

$$Mw_1(X_1) + w_2(X_2) \geq Mt_1 + M + w_2(X_2) > Mt_1,$$

again contradicting (7). We infer that $w_1(X_1) = t_1$ and consequently $X_1 \in \mathcal{E}_1$. Equation (7) together with $w_1(X_1) = t_1$ implies that $w_2(X_2) = 0$. Since w_2 is strictly positive on all V_2 , it follows that $X_2 = \emptyset$. Therefore, we have $X = \{z\} \cup X_1 \in \mathcal{E}$.

Now, suppose that $z \notin X$. In this case,

$$Mw_1(V_1) + t_2 = t = w(X) = w(X_1) + w(X_2) = Mw_1(X_1) + w_2(X_2),$$

which implies

$$Mw_1(X_1) + w_2(X_2) = Mw_1(V_1) + t_2. \quad (8)$$

We must have $X_1 = V_1$ since if there exists a vertex $v \in V_1 \setminus X_1$, then we would have

$$\begin{aligned} Mw_1(X_1) + w_2(X_2) &\leq Mw_1(V_1) - Mw_1(v) + w_2(X_2) \\ &\leq Mw_1(V_1) - M + w_2(X_2) \\ &< Mw_1(V_1) + t_2, \end{aligned}$$

where the last inequality follows from (6) and $t_2 \geq 0$. Therefore, equality (8) simplifies to $w_2(X_2) = t_2$, and consequently there exists a hyperedge $X_2 \in \mathcal{E}_2$. This implies that $X = V_1 \cup X_2 \in \mathcal{E}$.

For the converse direction, suppose that X is a subset of V such that $X \in \mathcal{E}$. We need to show that $w(X) = t$. We again consider two cases depending on whether $z \in X$ or not. Suppose first that $z \in X$. Then $X = \{z\} \cup X_1$ for some $X_1 \in \mathcal{E}_1$. Due to the property of w_1 , we have $w_1(X_1) = t_1$. Consequently,

$$\begin{aligned} w(X) &= w(z) + w(X_1) \\ &= M(w_1(V_1) - t_1) + t_2 + Mw_1(X_1) \\ &= Mw_1(V_1) - Mt_1 + t_2 + Mt_1 \\ &= Mw_1(V_1) + t_2 = t. \end{aligned}$$

Suppose now that $z \notin X$. Then $X = V_1 \cup X_2$ for some $X_2 \in \mathcal{E}_2$. Due to the property of w_2 , we have $w_2(X_2) = t_2$. Consequently,

$$\begin{aligned} w(X) &= w(V_1) + w(X_2) \\ &= Mw_1(V_1) + w_2(X_2) \\ &= Mw_1(V_1) + t_2 = t. \end{aligned}$$

This shows that we have $w(X) = t$ whenever $X \in \mathcal{E}$, and completes the proof. \square